§1.8 Continuity

Key points: ① Graphical meaning of continuity (discontinuity)

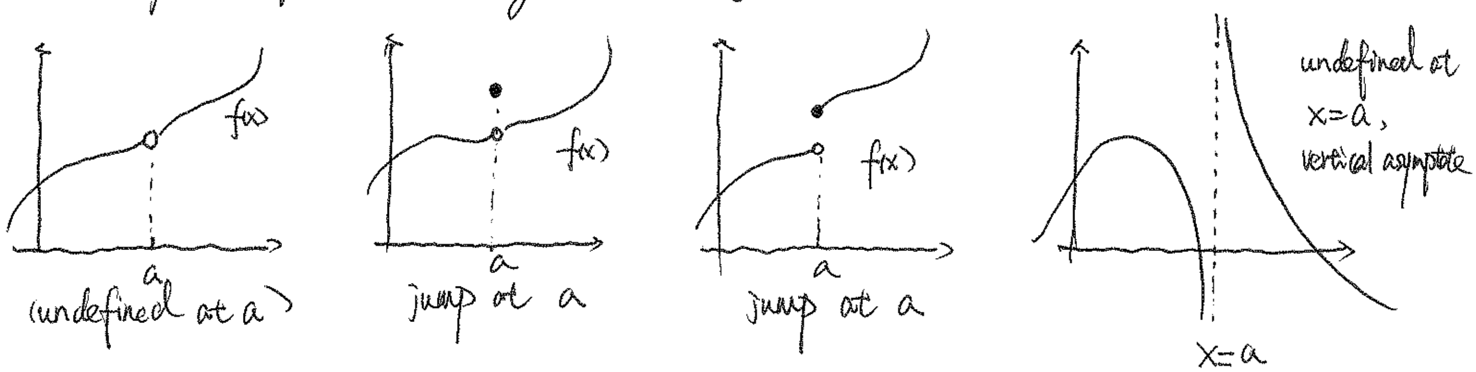
★ ② Continuity of piecewise function

★ ③ Intermediate Value Theorem.

Definition: $f(x)$ is continuous at $x=a$ if $\lim_{x\to a} f(x) = f(a)$.

Graphically, it means the graph of $y=f(x)$ passes through $(a, f(a))$ without jumps/holes.

eg.1 (Examples of discontinuity). The following functions are DISCONTINUOUS at $x=a$.



• Remark. Almost all functions with precise formulas are continuous except where undefined.

eg. 2. (S16). Let $g(x)=\begin{cases} x^3+2x & \text{if } x\le 5 \\ \dfrac{5x^2-x^3}{x-5} & \text{if } x>5 \end{cases}$. Is $g(x)$ continuous at $x=5$ or not? Why?

Remark: $g(5) = 5^3+2\cdot 5 = 135$. We need to find $\lim_{x\to 5} g(x)$ (exists or D.N.E) first.

Solution: $\lim_{x\to 5^-} g(x) = 5^3+2\cdot 5 = 135$ ($g(x)$ has left limit at $x=5$, which equals $g(5)$)

$\lim_{x\to 5^+} g(x) = \lim_{x\to 5^+} \dfrac{5x^2-x^3}{x-5} = \lim_{x\to 5^+} \dfrac{x^2\cdot(5-x)}{x-5}$ ('canceling zeros' trick in §1.6)

$= \lim_{x\to 5^+} -x^2 = -5^2 = -25$. (Plug in)

therefore, $\lim_{x\to 5^-} g(x) = 135 \neq -25 = \lim_{x\to 5^+} g(x)$. $\lim_{x\to 5} g(x)$ does not exist.

Therefore, $g(x)$ is not continuous at $x=5$.

eg 3. For what value of k will $f(x)=\begin{cases} \frac{x^2-3k}{x-3} & \text{if } x\le 2 \\ 8x-k & \text{if } x>2 \end{cases}$ be continuous for all x?
(F16)

Remark: $f(x)$ has presice expression piecewisely. The only point in issue is $\boxed{x=2}$.

Solution: In order to make $f(x)$ continuous at 2. We need to ensure $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$.

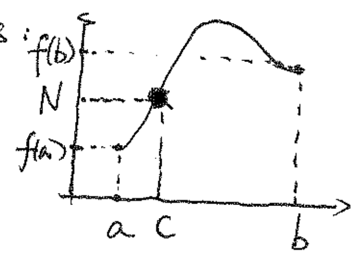
$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-} \frac{x^2-3k}{x-3} = \frac{4-3\cdot k}{2-3} = \frac{4-3k}{-1} = 3k-4$. and $f(2)=\frac{2^2-3k}{2-3}=3k-4$.

$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+} 8x-k = 16-k$. We need $\boxed{3k-4=16-k}$.

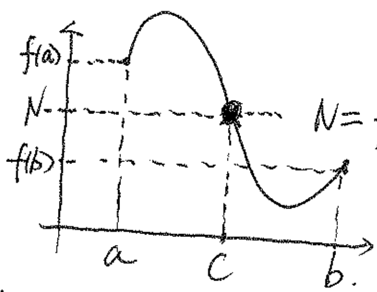
Solve for k, we have $4k=20 \Rightarrow \boxed{k=5}$.

☆ Intermediate Value Theorem (I.V.T. in exam formula sheet).

- If f is continuous on $[a,b]$, $f(a)\neq f(b)$, and N is between $f(a)$ & $f(b)$ then there exists $c\in(a,b)$ that satisfies $f(c)=N$.
- Graphically, IVT means:



$N=f(c)$. or



$N=f(c)$

☆ eg 4. Use the IVT to show that there is a solution to the equation $\cos x=\sqrt{x}$.
(F16)

Remark: In order to apply IVT, we need to pick $f(x)$, N, and the interval $[a,b]$.

W.T.S.: $\cos x=\sqrt{x} \iff \underbrace{\cos x-\sqrt{x}}_{f(x)} = \underbrace{0}_{N}$. $[a,b]$ should be some points where we can evaluate $\cos x$ like, 0, $\frac{\pi}{2}$ etc.

Solution: Let $f(x)=\cos x-\sqrt{x}$, $N=0$. $f(x)$ is continuous on $[0,+\infty)$.

And consider $f(x)$ on the interval $[0,\frac{\pi}{2}]$, i.e. $a=0$, $b=\frac{\pi}{2}$.

Then $f(0)=\cos 0-\sqrt{0}=1$, $f(\frac{\pi}{2})=\cos\frac{\pi}{2}-\sqrt{\frac{\pi}{2}}=-\sqrt{\frac{\pi}{2}}$. Hint: $\cos 0=1$, $\cos\frac{\pi}{2}=0$.

Therefore, $1>0>-\sqrt{\frac{\pi}{2}}$, i.e. $f(0)>N>f(\frac{\pi}{2})$ (N is between $f(0)$ and $f(\frac{\pi}{2})$)

According to IVT, there is $c\in(0,\frac{\pi}{2})$ such that $f(c)=N=0$, i.e.

$\cos c-\sqrt{c}=0 \iff \cos c=\sqrt{c}$.

§2.1. Derivatives

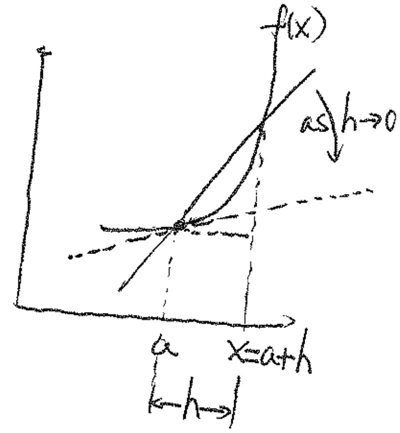
key points: ① Definition of Derivative: as a limit of average rate of change.

② Slope of tangent line as a derivative and the formula of tangent line.

③ Left and right derivatives of piecewise functions.

• Definition: The derivative of a function $f(x)$ at $x=a$, denoted $f'(a)$, given by:

$$\boxed{f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}}$$



• Remark 1: If we write $h = x - a \Leftrightarrow x = a+h$, then $h\to 0$ (h approaches 0) is equivalent to $x-a\to 0 \Leftrightarrow x\to a$ (x approaches a).

So $f'(a)$ can also be defined as

$$\boxed{f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}}$$

Remark 2: Notice that the above ratios are actually the average rate of change of the function over the interval $[a, a+h] = [a, x]$, which is the slope of the secant line. And the limit will be the slope of the tangent line passing through $(a, f(a))$, i.e., the slope of the tangent line $= f'(a)$.

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e.g. 1: Let $f(x) = \frac{1}{x+1}$. Find $f'(2)$ and the formula of the tangent line through $(2, f(2))$.

Solution: $f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{2+h+1} - \frac{1}{2+1}}{h}$ (simplify)

$$= \lim_{h\to 0} \frac{\frac{3-(3+h)}{(3+h)\cdot 3}}{h} = \lim_{h\to 0} \frac{-\cancel{h}}{(3+h)\cdot 3\cdot \cancel{h}}$$ Hint: $\frac{\frac{a}{b}}{c} = \frac{a}{b\cdot c}$

$$= \lim_{h\to 0} \frac{-1}{(3+h)\cdot 3} = \boxed{\frac{-1}{9}}$$ (Plug in $h=0$).

Notice $f(2) = \frac{1}{2+1} = \frac{1}{3}$.

Tangent line: through $(2, \frac{1}{3})$ with slope $f'(2) = -\frac{1}{9}$.

has formula: $\boxed{y - \frac{1}{3} = -\frac{1}{9}\cdot(x-2)}$

eg 2. Find $f'(0)$ for $f(x)=\sqrt{1-x}$.

Solution: $f'(0)=\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\dfrac{\sqrt{1-(0+h)}-\sqrt{1-0}}{h}$

$=\lim\limits_{h\to 0}\dfrac{\sqrt{1-h}-1}{h}$

Notice that if we plug in $h=0$, we have $\frac{0}{0}$. So we need to simplify the numerator first.

☆ (Conjugation Method).

$\sqrt{1-h}-1=(\sqrt{1-h}-1)\times\dfrac{\sqrt{1-h}+1}{\sqrt{1-h}+1}$

$=\dfrac{(\sqrt{1-h}-1)(\sqrt{1-h}+1)}{\sqrt{1-h}+1}=\dfrac{(\sqrt{1-h})^2-1^2}{\sqrt{1-h}+1}$

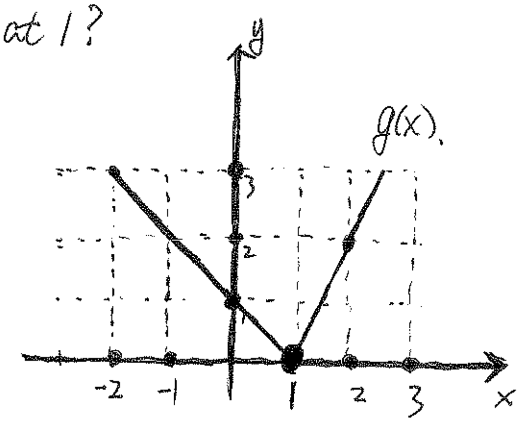
$=\dfrac{1-h-1}{\sqrt{1-h}+1}=\dfrac{-h}{\sqrt{1-h}+1}$

Remark: $\sqrt{A}+\sqrt{B}$ is called the conjugate radical of $\sqrt{A}-\sqrt{B}$. Notice that $(\sqrt{A}-\sqrt{B})(\sqrt{A}+\sqrt{B})=(\sqrt{A})^2-(\sqrt{B})^2=A-B$ helps remove the square root.

Therefore, $f'(0)=\lim\limits_{h\to 0}\dfrac{\frac{-h}{\sqrt{1-h}+1}}{h}=\lim\limits_{h\to 0}\dfrac{-1}{\sqrt{1-h}+1}=\boxed{\dfrac{-1}{2}}$ (Plug in $h=0$).

- $\lim\limits_{h\to 0^-}\dfrac{f(a+h)-f(a)}{h}$ and $\lim\limits_{h\to 0^+}\dfrac{f(a+h)-f(a)}{h}$ are called left and right derivatives of $f(x)$ at a.
- If $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ does not exist, we say $f(x)$ does not derivative at a.
- Linear function $y=k\cdot x+b$ has derivative k at any point, since all the secant lines are the same. The tangent line is the line itself.

eg 3. Given the graph of $g(x)$. Does $g(x)$ has derivative at 1?



Solution: Based on the geometric meaning of derivative, the left and Right derivates of $g(x)$ at $x=1$ are exactly the two slopes the two straight lines, i.e. $\lim\limits_{h\to 0^-}\dfrac{g(1+h)-g(1)}{h}=-1$

$\lim\limits_{h\to 0^+}\dfrac{g(1+h)-g(1)}{h}=2$

Left and right limits (derivatives) are not the same, therefore, $\lim\limits_{h\to 0}\dfrac{g(1+h)-g(1)}{h}$ D.N.E. $g(x)$ does not have derivative at $x=1$.

§2.2 Derivative Function

Key points: ① Compute $f'(x)$ via limit definition of derivative.

② Graph $f'(x)$ based on the graph of $f(x)$.

Replace a by x in $f'(a)$ and consider it as a new function of x. i.e.

Def: $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{z\to x} \frac{f(z)-f(x)}{z-h}$$ is called the derivative function.

Remarks:
- The domain of $f'(x)$ is where $f(x)$ has derivative.
- The process of computing $f'(x)$ is also called differentiate $f(x)$.
- If $f'(a)$ exists (at $x=a$), we say $f(x)$ is differentiable at a.
- We also have the following notations for derivative:

  $$f'(x) = \frac{df}{dx}, \quad f'(a) = \frac{df}{dx}\Big|_{x=a}$$
- Differentiable is stronger than continuous: If $f(x)$ is differentiable at a, then $f(x)$ is continuous at a, not vice versa.

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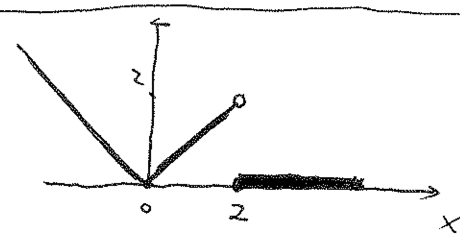
eg. 1. (3,6). True or Fasle: $f(x) = \begin{cases} |x| & \text{if } x<2 \\ 0 & \text{if } x \geq 2 \end{cases}$



(I) $f(x)$ is differentiable at $x=0$. (Fasle)

(II) $f(x)$ is continuous at $x=2$ (Fasle)

(III) $\lim_{x\to 0} f(x)$ exists.

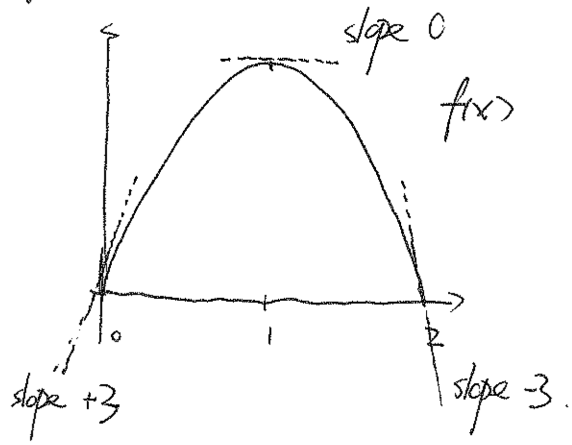
eg 2. Compute $r'(t)$ for $r(t) = 3t + \frac{5}{t}$.

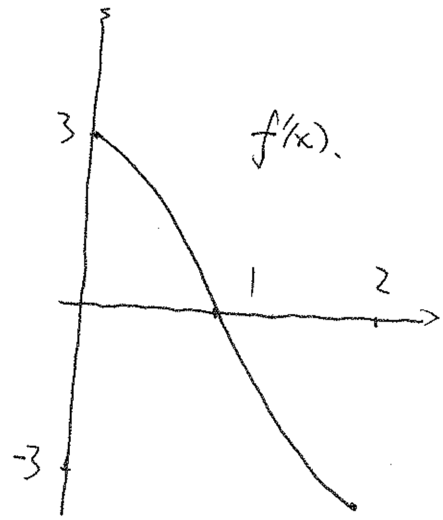
solution: $$r'(t) = \lim_{h\to 0} \frac{r(t+h)-r(t)}{h}$$ (think $r(t)$ as $f(x)$, where t plays the role of x)

$$= \lim_{h\to 0} \frac{\left[3(t+h) + \frac{5}{t+h}\right] - \left[3t + \frac{5}{t}\right]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3t}+3h\cancel{-3t} + \frac{5}{t+h} - \frac{5}{t}}{h} = \lim_{h\to 0} \frac{3h}{h} + \frac{\frac{5t-5t-5h}{(t+h)\cdot t}}{h} = \lim_{h\to 0} 3 - \frac{5}{(t+h)\cdot t} = \boxed{3 - \frac{5}{t^2}}$$

eg 3. For the function $f(x)$ shown below, sketch the graph of $f'(x)$.





eg 4 Consider $f(x)=\sqrt{1-2x}$. Find $f'(x)$ (via definition) and find an equation of a tangent line of $f(x)$ at $x=-4$.

(F16).

Solution: (a) $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\dfrac{\sqrt{1-2(x+h)}-\sqrt{1-2x}}{h}$. (conjugation method in §2.1)

$$\sqrt{1-2(x+h)}-\sqrt{1-2x}=\frac{\left(\sqrt{1-2(x+h)}-\sqrt{1-2x}\right)\cdot\left(\sqrt{1-2(x+h)}+\sqrt{1-2x}\right)}{\sqrt{1-2(x+h)}+\sqrt{1-2x}}$$

$$=\frac{[1-2(x+h)]-[1-2x]}{\sqrt{1-2(x+h)}+\sqrt{1-2x}}=\frac{-2h}{\sqrt{1-2(x+h)}+\sqrt{1-2x}}.$$

Therefore, $$f'(x)=\lim_{h\to 0}\frac{\dfrac{-2h}{\sqrt{1-2(x+h)}+\sqrt{1-2x}}}{h}=\lim_{h\to 0}\frac{-2}{\sqrt{1-2(x+h)}+\sqrt{1-2x}}\ \underset{h=0}{\overset{\text{Plug in}}{=}}\ \frac{-2}{\sqrt{1-2x}+\sqrt{1-2x}}$$

$$=\boxed{\frac{-1}{\sqrt{1-2x}}}$$

(b). slope of the tangent line at $x=-4$:

$$f'(-4)=\frac{-1}{\sqrt{1-2(-4)}}=\frac{-1}{\sqrt{1+8}}=\frac{-1}{\sqrt{9}}=\frac{-1}{3}.$$

passing through the point $(-4, f(-4))=(-4, \sqrt{1-2\cdot(-4)})=(-4, 3)$.

Equation: $y-3=-\frac{1}{3}\cdot(x-(-4))$

$$\boxed{y-3=-\frac{1}{3}\cdot(x+4)}$$